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Nonlinear sigma model Lagrangian for superfluid He3-A (B)

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Abstract. It is shown that the effective Lagrangian for the dynamics of superfluid He3 is described by the language of the $SU(2)$ chiral nonlinear sigma model in a unified way for A and B phases. The starting Lagrangian is assumed to be the nonlinear Schrödinger type. Here the key concept is to write the order parameter in terms of the rotation matrix from the intrinsic states and rewrite it with the equivalent $SU(2)$ matrix. The resultant effective Lagrangian is thus transcribed to the nonlinear sigma model for which the field takes the value on the $SU(2)$ manifold. The superfluid velocity of Mermin–Ho type is discussed in this representation and shown to be given by the topological quantity of $SU(2)$.

1. Introduction

The superfluid He3 is known to be an exotic form of matter due to the p-wave nature of the Cooper pair [1–4]†. It has two phases of different symmetric properties, A and B, with no magnetic field, and they are identified with the Anderson–Morel (ABM) and the Balian–Werthamer phases (BW) [5, 6]. Because of the p-wave nature of the interaction, the order parameter of He3 is described by the vector field $\Psi_{\mu i}$ with the orbital and spin vector indices ($\mu, i = 1, 2, 3$) in contrast to the ordinary superfluid He4 or s-wave superconductor with the scalar order parameter ψ. The p-wave nature is specifically revealed in the so-called anisotropic '$\boldsymbol{l}$-texture' of He3-A, where the order parameter is characterized by a special direction represented by the $\boldsymbol{l}$-vector (and also $\boldsymbol{d}$-vector in spin direction). In contrast to He3-A, the B phase is isotropic and has no such special direction, but it reveals some specific features that cannot be expected for the conventional superfluid.

These characteristic properties of the A and B phases of the superfluid He3 show up in the symmetry breaking patterns [2]. The superfluid He3 has both pure orbital and spin rotation symmetries $SO_l(3) \times SO_s(3)$ and the $U(1)$ gauge symmetry because of the small spin–orbital coupling. In the A phase, this symmetry is broken and the manifold of the internal degeneracy becomes $R_A = (S^2 \times SO(3))/Z_2$, where S_2 is a two-dimensional sphere for the $\boldsymbol{d}$-vector and $SO(3)$ for the solid rotation for the $\boldsymbol{l}$-texture. In the B phase, the degeneracy manifold becomes $R_B = S_1 \times SO_a(3)$: $S_1 \simeq U(1)$ is the phase degree of freedom and $SO_a(3)$ denotes the relative spin-to-orbital rotation.

As pointed by Volovik [2], the symmetry mentioned above is very similar to the chiral symmetry $SU_L(2) \times SU_R(2)$ for the quark in quantum chromodynamics (QCD). In the non-perturbative region of QCD, the chiral symmetry is considered to be broken into the isospin

† For reviews of the superfluid He3, see [1].

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symmetry $SU_V(2)$ and the degeneracy manifold becomes the axial $SU_A(2)$†. The dynamical origin of this symmetry breaking in QCD is still not clear, but a lot of low-energy hadronic properties can be explained as the low-energy theorem for this chiral symmetry breaking. When we consider the two-value equivalence $SO(3) \sim SU(2)/Z_2$, the He3 symmetry $SO_l(3) \times SO_s(3)$ is equivalent to the chiral symmetry $SU_L(2) \times SU_R(2)$, and, for the B phase, the symmetry breaking pattern is the same as the vacuum of QCD [7]. Because of that symmetry breaking pattern, the low-energy effective theory of QCD is known to be reduced to the $SU(2)$ nonlinear (NL) σ model with the $SU(2)$-matrix field $U(x)$ as the elementary degrees of freedom [8–10].

The similarity between the superfluid He3-B and the chiral symmetries suggests that the superfluid He3-B can be described by using the $SU(2)$ NL σ model and some topological properties of it can be derived from those of the $SU(2)$ group. The similarity in the symmetry breaking structure also exists between the He3-A and the Weinberg–Salam model for the electroweak interactions [2], where $SU_W(2) \times U_Y(1)$ symmetry is broken down to $U_{comb}(1)$: the combined symmetry with $e^{i\theta\tau_3}$ and $U_Y(1)$. Thus the residual manifold becomes $R_{WS} = SU_W(2) \times U_Y(1)/U_{comb}(1)$ which is equivalent to R_A in the He3-A with the two-value equivalence $SO(3) \sim SU(2)/Z_2$.

The purpose of this paper is to give a unified construction of the effective field Lagrangian for both superfluid He3-A and B phases in terms of the language of the $SU(2)$ NL σ model. The essence of our approach is based on the following two points. (a) The order parameter for He3-A and B is realized by rotating some initial vector, which accommodates the symmetry breaking. (b) The starting Landau–Ginzburg Lagrangian is given by a generalized version of the NL Schrödinger form which has been used for the ordinary superfluid written in terms of the scalar order parameter [11–13], that is

$$L = \int \left[\tfrac{1}{2} i(\Psi^* \dot{\Psi} - \mathrm{c.c.}) - \frac{1}{2m} |\nabla \Psi|^2 - V(|\Psi|) \right] d^2x \tag{1}$$

where m is twice the effective mass of the He3 atom and $V(|\Phi|)$ is the potential for the order parameter $\Psi = (\Psi_{\mu i})$.

This paper is organized as follows. In the next section, we give a general construction of the NL representation for the order parameter of superfluid He3 including ABM and BW as special cases. In section 3, we consider the explicit form for the effective Lagrangian of the superfluid He3-A and B, respectively, with replacing the order parameter Ψ by the unitary field $U(x)$. Some discussions for the topological properties are also given with that NL representation in the final section.

2. Nonlinear representation for the order parameter

2.1. Order parameter decomposition

The order parameter of He3 represented by the 3×3 matrix field, $\Psi(x) = (\Psi_{\mu i}(x))$, can be divided as $\Psi(x) = \phi(x)R(x)$ where $R(x) \in SO(3)$ and $\phi(x)$ represents a 3×3 Hermite matrix whose eigenvalues are positive or zero (polar or Iwasawa decomposition) [14]. The matrix field $\phi(x)$ plays the same role as the density scalar field $\rho(x)$ in the Ginzburg–Landau theory for He4 and its explicit form should be specified later for each phase of He3. As the

† For chiral symmetry breaking in QCD, see [7].

effective Lagrangian, we consider the canonical term L_C, the kinetic term T and the potential term V:

$$L_C = \tfrac{1}{2}\mathrm{i}\hbar\,\mathrm{Tr}(\Psi^\dagger\dot\Psi - \dot\Psi^\dagger\Psi) \qquad T = -\frac{\hbar^2}{2m}|\partial\Psi|^2 \qquad V = V(|\Psi|^2). \tag{2}$$

To rewrite (2) with ϕ and R, we use

$$\mathrm{Tr}\,\Psi^\dagger\dot\Psi = \mathrm{Tr}\,R^\dagger\phi^\dagger\{\dot\phi R + \phi\dot R\} = \mathrm{Tr}\,\phi^\dagger\dot\phi + \mathrm{Tr}\,g\dot R R^\dagger \tag{3a}$$

$$\mathrm{Tr}\,\dot\Psi^\dagger\Psi = \mathrm{Tr}\{\dot\phi^\dagger R^\dagger + \phi^\dagger\dot R^\dagger\}\phi R = \mathrm{Tr}\,\dot\phi^\dagger\phi - \mathrm{Tr}\,g\dot R R^\dagger \tag{3b}$$

$$|\partial\Psi|^2 = \mathrm{Tr}\,\partial\Psi^\dagger\partial\Psi = |\partial\phi|^2 + \mathrm{Tr}\,g\partial R\partial R^\dagger - 2\,\mathrm{Tr}(\phi^\dagger\partial\phi)(\partial R R^\dagger) \tag{3c}$$

$$|\Psi|^2 = \mathrm{Tr}\,\Psi^\dagger\Psi = |\phi|^2 \tag{3d}$$

where $g(x) = \phi^\dagger(x)\,\phi(x)$ plays the role of the metric field for the spin indices (and ϕ is the corresponding 'dreibein' field). By substituting equations (3*a*)–(3*d*) and (4*a*) into (2), we obtain

$$L_C = \tfrac{1}{2}\mathrm{i}\hbar\,\mathrm{Tr}(\phi^\dagger\dot\phi - \dot\phi^\dagger\phi) + \mathrm{i}\hbar\,\mathrm{Tr}\,g\dot R R^\dagger \tag{4a}$$

$$T = -\frac{\hbar^2}{2m}|\partial\phi|^2 - \frac{\hbar^2}{2m}\,\mathrm{Tr}\,g\partial R\partial R^\dagger + \frac{\hbar^2}{m}\,\mathrm{Tr}(\phi^\dagger\partial\phi)(\partial R R^\dagger) \tag{4b}$$

$$V = V(|\phi|^2). \tag{4c}$$

The second term in equation (4*b*) can be rewritten as

$$\mathrm{Tr}\,g\partial R\partial R^\dagger = -\,\mathrm{Tr}\,g(\partial R R^\dagger)^2 \tag{5}$$

thus the $R(x)$ field appears only in the form $\mathrm{d}R\,R^\dagger$ in the effective Lagrangian.

2.2. Unitary field description

To transform the effective Lagrangian (4*a*)–(4*c*) into the form of the $SU(2)$ NL σ model, we use the relation

$$R_{ai} = \tfrac{1}{2}\,\mathrm{Tr}\,U^\dagger\tau_a U\tau_i = \tfrac{1}{2}\,\mathrm{Tr}\,\tau_a U\tau_i U^\dagger \tag{6}$$

where $U = U(x) \in SU(2)$ is a unitary matrix field and τ_a is the Pauli matrix. Equation (6) represents the two-valued equivalence $SO(3) \simeq SU(2)/Z_2$. Using equation (6), $\partial R\,R^\dagger$ becomes†

$$(\partial_\mu R R^\dagger)_{ab} = 2L^\alpha_\mu\epsilon_{\alpha ab} \tag{7}$$

where L^α_μ is the Maurer–Cartan form of $SU(2)$:

$$\mathrm{d}U\,U^\dagger = \mathrm{i}L^\alpha_\mu\tau_\alpha\,\mathrm{d}x^\mu. \tag{8}$$

Substituting equation (7) into equations (4*a*)– (4*c*), we obtain the general form of the effective Lagrangian in the $SU(2)$ NL σ form:

$$L_C = \tfrac{1}{2}\mathrm{i}\hbar\,\mathrm{Tr}(\phi^\dagger\dot\phi - \dot\phi^\dagger\phi) - \mathrm{i}\hbar A_\alpha(g)L^\alpha_0 \tag{9a}$$

$$T = -\frac{\hbar^2}{2m}|\partial\phi|^2 - \frac{2\hbar^2}{m}\left[\rho(L^a_i)^2 - g_{ab}L^a_iL^b_i\right] - \frac{\hbar^2}{m}A_\alpha(\phi^\dagger\partial_i\phi)L^\alpha_i \tag{9b}$$

$$V = V(|\phi|^2) \tag{9c}$$

† See appendix A for the derivations of equations (7) and (9*a*)–(9*c*).

where $\rho(x) = \mathrm{Tr}\, g(x)$ and A_α is the operation for the Hodge-star operator [15] defined by

$$A_\alpha(M) = \tfrac{1}{2}\epsilon_{\alpha ab} M_{ab} \tag{10}$$

for any 3×3 matrices. It should be noticed that the term $(L_i^a)^2$ in (9*b*) gives nothing but the canonical term in the $SU(2)$ NL σ model:

$$(L_\mu^a)^2 = \tfrac{1}{2}\,\mathrm{Tr}\,\partial_\mu U \partial_\mu U^\dagger. \tag{11}$$

3. Superfluid phases of He3

Following the general consideration of the group structure of the order parameter for the superfluid He3, we shall derive the effective Lagrangian for the A and B phases, respectively.

3.1. Balian–Werthamer phase (BW)

The BW phase is the case where $\phi(x) = \Delta_B(x) I$, where $I = (\delta_{ij})$ is a 3×3 unit matrix. Then the quantities defined in section 2 are given by

$$g = |\Delta_B|^2 I \qquad \rho = 3|\Delta_B|^2 \qquad \phi^\dagger \partial\phi = \Delta_B^* \partial \Delta_B I \qquad A(g) = 0 \qquad A(\phi^\dagger\partial\phi) = 0. \tag{12}$$

Using equation (12) for equations (9*a*)–(9*c*), we obtain the effective Lagrangian for the BW-phase:

$$L_C = \tfrac{3}{2}\mathrm{i}\hbar(\Delta_B^* \dot{\Delta}_B - \dot{\Delta}_B^* \Delta_B) \tag{13a}$$

$$T = -\tfrac{3}{2}\mathrm{i}\hbar|\partial\Delta_B|^2 - \frac{4\hbar^2}{m}\sum_{a=1}^{3}(L_i^a)^2 \tag{13b}$$

$$V = V(|\Delta_B|^2). \tag{13c}$$

For the uniform superfluid ($\partial\Delta_B = 0$), the kinetic term T in equation (13*b*) is completely the same as the $SU(2)$ NL σ model:

$$T = -\frac{4\hbar^2}{m}\sum_{a=1}^{3}(L_i^a)^2 = \frac{2\hbar^2}{m}\,\mathrm{Tr}\,\partial U \partial U^\dagger \tag{14}$$

which shows that the same symmetry breaking pattern as the chiral symmetry of QCD is realized for the BW-phase of the superfluid He3.

3.2. Anderson–Morel phase (ABM)

The ABM phase can be described with the order parameter

$$\Psi_{\mu i} = \Delta_A d_\mu \psi_i \qquad \psi = \mathrm{e}^{-\mathrm{i}\gamma}(\boldsymbol{m} + \mathrm{i}\boldsymbol{n}) \tag{15}$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are the orthogonal unit vectors parametrized by

$${}^t\boldsymbol{m} = (\cos\beta\cos\alpha, \cos\beta\sin\alpha, -\sin\beta) \qquad {}^t\boldsymbol{n} = (-\sin\alpha, \cos\alpha, 0). \tag{16}$$

It should be noticed that ψ can be written as

$$\psi = R_z(\alpha)R_y(\beta)R_z(\gamma)\boldsymbol{v} \qquad {}^t\boldsymbol{v} = (1, i, 0) \tag{17}$$

where R_z and R_y are the rotation matrices around the z and y axes:

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

In the derivation of equation (17), we have used the special property of $\boldsymbol{v}$ under the gauge symmetry $\mathrm{e}^{-\mathrm{i}\gamma}\boldsymbol{v} = R_z(\gamma)\boldsymbol{v}$. The rotation matrix $R_z(\alpha)R_y(\beta)R_z(\gamma)$ in (17) just gives the Euler-angle representation of the $SO(3)$ element [16], so that the ABM phase can also be represented in our formalism: $\phi = \boldsymbol{d} \otimes \Delta_A \boldsymbol{v} = \Delta_A(\boldsymbol{d} \otimes \boldsymbol{v})$ and $R = (R_z(\alpha)R_y(\beta)R_z(\gamma))^{\dagger}$. Then we obtain

$$g = |\Delta_A|^2 \boldsymbol{v}^{\dagger} \otimes \boldsymbol{v} \qquad \rho = 2|\Delta_A|^2 \qquad A(g) = \mathrm{i}|\Delta_A|^2 \delta_{3,\alpha} \tag{19}$$

where

$$\boldsymbol{v}^{\dagger} \otimes \boldsymbol{v} = \begin{pmatrix} 1 & \mathrm{i} & 0 \\ -\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Using equation (19), we obtain the effective Lagrangian for the ABM phase:

$$L_C = \mathrm{i}\hbar(\Delta_A^* \partial \Delta_A - \partial \Delta_A^* \Delta_A) - \hbar |\Delta_A|^2 L_0^3 \tag{21a}$$

$$T = -\frac{\hbar^2}{m}|\partial(\Delta_A \boldsymbol{d})|^2 - \frac{2\hbar^2}{m}|\Delta_A|^2[(L_i^1)^2 + (L_i^2)^2 + 2(L_i^3)^2] - \frac{\mathrm{i}\hbar^2}{m}(\Delta_A^* \partial_i \Delta_A) L_i^3 \tag{21b}$$

$$V = V(|\Delta_A|). \tag{21c}$$

For the uniform superfluid, equations (21a) and (21b) give

$$L_C = -\hbar|\Delta_A|^2 L_0^3 \qquad T = -\frac{2\hbar^2}{m}|\Delta_A|^2[(L_i^1)^2 + (L_i^2)^2 + 2(L_i^2)^2] \tag{22}$$

where the second term of T shows the asymmetry for the z-direction in the ABM-phase and the second term of L_C is the same Wess–Zumino term [17] that was discussed for the system of an electron around a magnetic monopole [18] and in the $SU(3)$ Skyrme model [19]†.

We should also mention the similarity and the differences between the ABM phase and the Weinberg–Salam model here. The Higgs field $\phi = (\phi_1, \phi_2)$ of the Weinberg–Salam model is invariant under $SU(2) \times U(1)$ symmetry. For the normal vacuum, this field is spontaneously broken into $\phi_0 = (0, v)$ that has a similar gauge symmetry $\mathrm{e}^{-\mathrm{i}\gamma}\phi_0 = \mathrm{e}^{\mathrm{i}(\gamma/2)\tau_3}\phi_0$ to the ABM phase of the superfluid He3. However, because of the different spin magnitudes (1 for the ABM phase and $\frac{1}{2}$ for the Weinberg–Salam model) and the existence of the gauge field and the Higgs mechanism in the Weinberg–Salam theory, the nonlinear sigma model for the Weinberg–Salam model becomes somewhat different from (22).

4. Topological quantities

In this section, we take a typical example that reveals the topological properties of the superfluid He3. This feature shows up naturally in the NL representation of the effective Lagrangian.

The most fundamental topological quantities of $SU(2)$ are the left and right Maurer–Cartan forms:

$$\mathrm{d}U\, U^{\dagger} = \mathrm{i}L^a \tau_a = \mathrm{i}L_i^a\, \mathrm{d}\theta_i \tau_a \qquad U^{\dagger}\, \mathrm{d}U = \mathrm{i}R^a \tau_a = \mathrm{i}R_i^a\, \mathrm{d}\theta_i\, \tau_a \tag{23}$$

† The Wess–Zumino term has also been discussed in the superfluid He3 about the vortex motion [4, 20–22]. See also [23].

where $U \in SU(2)$. Let us consider the ABM-phase, and the rotation matrix field R is parametrized as $R(\alpha, \beta, \gamma) = (R_z(\alpha)R_y(\beta)R_z(\gamma))^\dagger = R_z(-\gamma)R_y(-\beta)R_z(-\alpha)$, consistent with equation (17). By carrying out a direct calculation using equation (6), the unitary matrix corresponding to $R(\alpha, \beta, \gamma)$ is shown to be $U(\alpha, \beta, \gamma) = U_z(-\gamma)U_y(-\beta)U_z(-\alpha)$ and, this is just the Euler angle parametrization of the rotation matrix. With this Euler angle parametrization, the Maurer–Cartan forms are given by†

$$L^1 = \tfrac{1}{2}\mathrm{i}[\sin\gamma\,\mathrm{d}\beta - \sin\beta\cos\gamma\,\mathrm{d}\alpha] \tag{24a}$$

$$L^2 = \tfrac{1}{2}\mathrm{i}[\cos\gamma\,\mathrm{d}\beta + \sin\beta\sin\gamma\,\mathrm{d}\alpha] \tag{24b}$$

$$L^3 = \tfrac{1}{2}\mathrm{i}[\mathrm{d}\gamma + \cos\beta\,\mathrm{d}\alpha] \tag{24c}$$

together with

$$R^1 = \tfrac{1}{2}\mathrm{i}[\sin\alpha\,\mathrm{d}\beta - \sin\beta\cos\alpha\,\mathrm{d}\gamma] \tag{25a}$$

$$R^2 = \tfrac{1}{2}\mathrm{i}[-\cos\alpha\,\mathrm{d}\beta - \sin\beta\sin\alpha\,\mathrm{d}\gamma] \tag{25b}$$

$$R^3 = \tfrac{1}{2}\mathrm{i}[\mathrm{d}\alpha + \cos\beta\,\mathrm{d}\gamma]. \tag{25c}$$

To discuss the physical interpretation, we consider the symmetry of the ABM-phase. As is clear from equations (21*a*)–(21*c*), the effective Lagrangian for the ABM-phase is invariant under the limited chiral symmetries:

$$U_L(1): \quad U(\alpha, \beta, \gamma) \longrightarrow U_z(\theta)R(\alpha, \beta, \gamma) \tag{26a}$$

$$SU_R(2): \quad U(\alpha, \beta, \gamma) \longrightarrow U(\alpha, \beta, \gamma)g^\dagger \tag{26b}$$

where $g \in SU(2)$. $SU_L(2)$ corresponds to the orbital rotation symmetry $SO_l(3)$, and $U_R(1)$ to the gauge symmetry, so that the invariance of the Lagrangian under these symmetries is clear physically. The conserved currents for these symmetries can be calculated using Noether's theorem, and we obtain L_i^3 for $U_L(1)$ and R_i^a for $SU_R(2)$. L_i^3 is just the current for the gauge transformation of the order parameter, so it should be proportional to the superfluid velocity. Using equation (24*c*) and properly fixed constants, the explicit form of this becomes

$$\boldsymbol{v} = -\frac{\hbar}{m}\boldsymbol{L}^3 = -\frac{\hbar}{2m}(\nabla\gamma + \cos\beta\nabla\alpha). \tag{27}$$

This is just the superfluid velocity given by Mermin and Ho [24]. The vorticity $\omega = \mathrm{d}v$ is given by

$$\omega = \frac{\hbar}{m}\,\mathrm{Tr}\,\tau_3\boldsymbol{L}^2 \tag{28}$$

where the Maurer–Cartan relation $\mathrm{d}\boldsymbol{L} - \boldsymbol{L}^2 = 0$ has been used.

5. Summary

We have shown that the effective Lagrangian of the superfluid He3 can be represented in terms of the $SU(2)$ NL σ model for any phases in a unified manner. In particular, we have given the explicit form of the Lagrangian for the ABM and BW phases. The topological structures of the superfluid He3 can be related to the topological properties of $SU(2)$ in this representation. As one example, we have shown that the Mermin–Ho velocity can be understood as the Maurer–Cartan form of $SU(2)$. The topological properties of the superfluid He3 typically appear in vortices [4, 22]. Thus, it will be interesting to study the role of topological properties for the motion of superfluid vortices with the $SU(2)$ representation developed in this paper.

† See appendix B for a derivation of equations (24*a*)–(24*c*) and (25*a*)–(25*c*).

Appendix A. Derivation of Maurer–Cartan-form representation

Using equation (6), we can easily obtain

$$R_{ai}\tau_i = U^\dagger \tau_a U \qquad R_{ai}\tau_a = U\tau_i U^\dagger. \tag{A1}$$

Let us calculate the $\partial R R^\dagger$. The differentiation of equation (A1) gives

$$\partial R_{ai}\tau_i = \partial U^\dagger \tau_a U + U^\dagger \tau_a \partial U. \tag{A2}$$

Multiplying equation (A2) by U and $U^\dagger$ from both sides, we obtain

$$\partial R_{ai} R_{bi}\tau_b = \partial R_{ai} U\tau_i U^\dagger = [\tau_a, \partial U U^\dagger] \tag{A3}$$

where $\partial U U^\dagger + U\partial U^\dagger = 0$ has been used. The Maurer–Cartan form L^a_μ for U is defined by

$$\mathrm{d}U\, U^\dagger = \mathrm{i}L^\alpha_\mu \tau_\alpha\, \mathrm{d}x^\mu. \tag{A4}$$

On substituting equation (A4) into (A3), we obtain equation (7):

$$(\partial_\mu R R^\dagger)_{ab} = 2L^\alpha_\mu \epsilon_{\alpha ab}. \tag{A5}$$

The derivations of equations (9*a*)–(9*c*), can be done easily with the formulae

$$\mathrm{Tr}\, g\dot{R}R^\dagger = g_{ab}(\dot{R}R^\dagger)_{ba} = -2\epsilon_{\alpha ab} g_{ab} L^\alpha_0 = -A_\alpha(g) L^\alpha_0 \tag{A6a}$$

$$\mathrm{Tr}\, g\partial R\partial R^\dagger = -g_{ab}(\partial R R^\dagger)_{bc}(\partial R R^\dagger)_{ca} = -4\epsilon_{\alpha bc}\epsilon_{\beta ca} L^\alpha_i L^\beta_i = 4\eta L^a_i L^a_i - 4g_{ab} L^a_i L^b_i \tag{A6b}$$

$$\mathrm{Tr}(\phi^\dagger\partial\phi)(\partial R R^\dagger) = -2\epsilon_{\alpha ab}(\phi^\dagger\partial_i\phi)_{ab} L^\alpha_i = -A_\alpha(\phi^\dagger\partial_i\phi) L^\alpha_i \tag{A6c}$$

where $\rho = \mathrm{Tr}\, g$ and A_α is the operation for the Hodge-star operator and is defined by equation (10).

Appendix B. Euler-angle representation of the Maurer–Cartan form

The unitary transformation corresponding to $R_i(\theta)$ is

$$U_i(\theta) = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta\tau_i} = \cos\tfrac{1}{2}\theta + \mathrm{i}\tau_i \sin\tfrac{1}{2}\theta \tag{B1}$$

which can be checked directly using equation (6). Differentiating the Euler-angle representation $U(\alpha, \beta, \gamma) = U_z(-\gamma)U_y(-\beta)U_z(-\alpha)$, we obtain

$$\mathrm{d}U = \mathrm{d}U_z(-\gamma)\, U_y(-\beta)U_z(-\alpha) + U_z(-\gamma)\, \mathrm{d}U_y(-\beta)\, U_z(-\alpha) + U_z(-\gamma)U_y(-\beta)\, \mathrm{d}U_z(-\alpha) \tag{B2}$$

so that

$$\mathrm{d}U\, U^\dagger = \mathrm{d}U_z(-\gamma)\, U^\dagger_z(-\gamma) + U_z(-\gamma)[\mathrm{d}U_y(-\beta)\, U^\dagger_y(-\beta)]U^\dagger_z(-\gamma)$$
$$+U_z(-\gamma)U_y(-\beta)[\mathrm{d}U_z(-\alpha)\, U^\dagger_z(-\alpha)]U^\dagger_y(-\beta)U^\dagger_z(-\gamma). \tag{B3}$$

For the evaluation of equation (B3), we use

$$\mathrm{d}U_i(-\theta)\, U^\dagger_i(-\theta) = \tfrac{1}{2}\mathrm{i}\, \mathrm{d}\theta\, \tau_i \tag{B4}$$

and

$$U_z(-\gamma)\tau_2 U^\dagger_z(-\gamma) = \cos\gamma\,\tau_2 + \sin\gamma\,\tau_3 \tag{B5a}$$

$$U_z(-\gamma)U_y(-\beta)\tau_3 U^\dagger_y(-\beta)U^\dagger_y(-\gamma) = -\cos\gamma\,\sin\beta\,\tau_1 + \sin\gamma\,\sin\beta\,\tau_2 + \cos\beta\,\tau_3. \tag{B5b}$$

Calculating equation (B5*a*) directly, we obtain the left Maurer–Cartan form L^a in equations (24*a*)–(24*c*). The right Maurer–Cartan form in equations (25*a*)–(25*b*) is also calculated in a similar way.

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